Now, by assumption, the sphere is small enough that $\mathbf{P}$ does not vary significantly over its volume, so the term *left out* of the integral in Eq. 4.17 corresponds to the field at the center of a *uniformly* polarized sphere, to wit: $-(1/3\epsilon_0)\mathbf{P}$ (Eq. 4.14). But this is precisely what $\mathbf{E}_{\text{in}}$ (Eq. 4.18) puts back in! The macroscopic field, then, is given by the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau', \tag{4.19}$$

where the integral runs over the *entire* volume of the dielectric. This is, of course, what we used in Sect. 4.2.1; without realizing it, we were correctly calculating the averaged, macroscopic field, for points inside the dielectric.

You may have to reread the last couple of paragraphs for the argument to sink in. Notice that it all revolves around the curious fact that the average field over *any* sphere (due to the charge inside) is the same as the field at the center of a *uniformly polarized* sphere with the same total dipole moment. This means that no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field. Incidentally, while the argument ostensibly relies on the spherical shape I chose to average over, the macroscopic field is certainly independent of the geometry of the averaging region, and this is reflected in the final answer, Eq. 4.19. Presumably one could reproduce the same argument for a cube or an ellipsoid or whatever—the calculation might be more difficult, but the conclusion would be the same.

4.3 ■ THE ELECTRIC DISPLACEMENT

4.3.1 ■ Gauss's Law in the Presence of Dielectrics

In Sect. 4.2 we found that the effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\nabla\cdot\mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything *else* (which, for want of a better term, we call **free charge**, ρ_f). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is *not* a result of polarization. Within the dielectric, the total charge density can be written:

$$\rho = \rho_b + \rho_f, \tag{4.20}$$

and Gauss's law reads

$$\epsilon_0\nabla\cdot\mathbf{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\mathbf{P} + \rho_f,$$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

It is convenient to combine the two divergence terms:

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$$

The expression in parentheses, designated by the letter **D**,

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},} \tag{4.21}$$

is known as the **electric displacement**. In terms of **D**, Gauss's law reads

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f,} \tag{4.22}$$

or, in integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}, \tag{4.23}$$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because *it makes reference only to free charges,* and free charge is the stuff we control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues, by the mechanisms of Sect. 4.1, and this polarization produces the bound charge. In a typical problem, therefore, we know ρ_f, but we do not (initially) know ρ_b; Eq. 4.23 lets us go right to work with the information at hand. In particular, whenever the requisite symmetry is present, we can immediately calculate **D** by the standard Gauss's law methods.

Example 4.4. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.

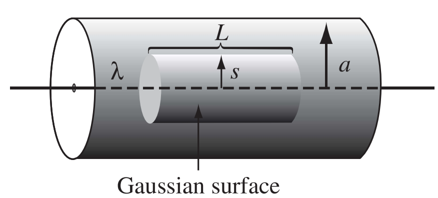

FIGURE 4.17

Solution

Drawing a cylindrical Gaussian surface, of radius s and length L, and applying Eq. 4.23, we find

$$D(2\pi s L) = \lambda L.$$

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}. \tag{4.24}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know **P**.

It may appear to you that I left out the surface bound charge σ_b in deriving Eq. 4.22, and in a sense that is true. We cannot apply Gauss's law precisely *at* the surface of a dielectric, for here ρ_b blows up,[6] taking the divergence of **E** with it. But everywhere *else* the logic is sound, and in fact if we picture the edge of the dielectric as having some finite thickness, within which the polarization tapers off to zero (probably a more realistic model than an abrupt cut-off anyway), then there *is* no surface bound charge; ρ_b varies rapidly but smoothly within this "skin," and Gauss's law can be safely applied *everywhere.* At any rate, the integral form (Eq. 4.23) is free from this "defect."

Problem 4.15 A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\,\hat{\mathbf{r}},$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

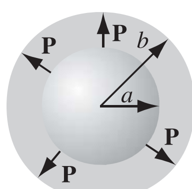

FIGURE 4.18

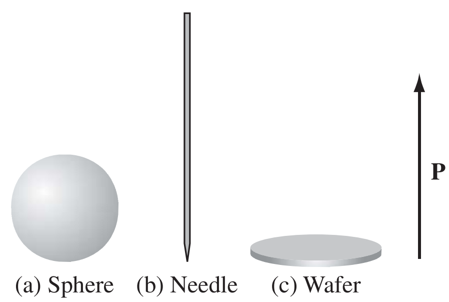

FIGURE 4.19

[6]The polarization drops abruptly to zero outside the material, so its *derivative* is a delta function (see Prob. 1.46). The surface bound charge *is* precisely this term—in this sense it is actually *included* in ρ_b, but we ordinarily prefer to handle it separately as σ_b.

(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find $\mathbf{D}$, and then get $\mathbf{E}$ from Eq. 4.21. [Notice that the second method is much faster, and it avoids any explicit reference to the bound charges.]

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is $\mathbf{D}_0 = \epsilon_0\mathbf{E}_0 + \mathbf{P}$.

(a) Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and $\mathbf{P}$. Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and $\mathbf{P}$. Assume the polarization is "frozen in," so it doesn't change when the cavity is excavated.

(b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{P}$ (Fig. 4.19b).

(c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{P}$ (Fig. 4.19c).

Assume the cavities are small enough that $\mathbf{P}$, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

4.3.2 ■ A Deceptive Parallel

Equation 4.22 looks just like Gauss's law, only the *total* charge density ρ is replaced by the *free* charge density ρ_f, and $\mathbf{D}$ is substituted for $\epsilon_0\mathbf{E}$. For this reason, you may be tempted to conclude that $\mathbf{D}$ is "just like" $\mathbf{E}$ (apart from the factor ϵ_0), except that its source is ρ_f instead of ρ: "To solve problems involving dielectrics, you just forget all about the bound charge—calculate the field as you ordinarily would, only call the answer $\mathbf{D}$ instead of $\mathbf{E}$." This reasoning is seductive, but the conclusion is false; in particular, there is no "Coulomb's law" for $\mathbf{D}$:

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho_f(\mathbf{r}')\,d\tau'.$$

The parallel between $\mathbf{E}$ and $\mathbf{D}$ is more subtle than that.

For the divergence alone is insufficient to determine a vector field; you need to know the curl as well. One tends to forget this in the case of electrostatic fields because the curl of $\mathbf{E}$ is always zero. But the curl of $\mathbf{D}$ is *not* always zero.

$$\boldsymbol{\nabla} \times \mathbf{D} = \epsilon_0(\boldsymbol{\nabla} \times \mathbf{E}) + (\boldsymbol{\nabla} \times \mathbf{P}) = \boldsymbol{\nabla} \times \mathbf{P}, \tag{4.25}$$

and there is no reason, in general, to suppose that the curl of $\mathbf{P}$ vanishes. Sometimes it does, as in Ex. 4.4 and Prob. 4.15, but more often it does not. The bar electret of Prob. 4.11 is a case in point: here there is no free charge anywhere, so if you really believe that the only source of $\mathbf{D}$ is ρ_f, you will be forced to conclude that $\mathbf{D} = \mathbf{0}$ everywhere, and hence that $\mathbf{E} = (-1/\epsilon_0)\mathbf{P}$ inside and $\mathbf{E} = \mathbf{0}$ outside the electret, which is obviously wrong. (I leave it for you to find the place where $\boldsymbol{\nabla} \times \mathbf{P} \neq \mathbf{0}$ in this problem.) Because $\boldsymbol{\nabla} \times \mathbf{D} \neq \mathbf{0}$, moreover, $\mathbf{D}$ cannot be expressed as the gradient of a scalar—there is no "potential" for $\mathbf{D}$.

Advice: When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, then you can get **D** directly from Eq. 4.23 by the usual Gauss's law methods. (Evidently in such cases $\nabla \times \mathbf{P}$ is automatically zero, but since symmetry alone dictates the answer, you're not really obliged to worry about the curl.) If the requisite symmetry is absent, you'll have to think of another approach, and, in particular, you must *not* assume that **D** is determined exclusively by the free charge.

4.3.3 ■ Boundary Conditions

The electrostatic boundary conditions of Sect. 2.3.5 can be recast in terms of **D**. Equation 4.23 tells us the discontinuity in the component perpendicular to an interface:

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f, \tag{4.26}$$

while Eq. 4.25 gives the discontinuity in parallel components:

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}. \tag{4.27}$$

In the presence of dielectrics, these are sometimes more useful than the corresponding boundary conditions on **E** (Eqs. 2.31 and 2.32):

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma, \tag{4.28}$$

and

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}. \tag{4.29}$$

You might try applying them, for example, to Probs. 4.16 and 4.17.

Problem 4.17 For the bar electret of Prob. 4.11, make three careful sketches: one of **P**, one of **E**, and one of **D**. Assume L is about $2a$. [*Hint:* **E** lines terminate on charges; **D** lines terminate on *free* charges.]

4.4 ■ LINEAR DIELECTRICS

4.4.1 ■ Susceptibility, Permittivity, Dielectric Constant

In Sects. 4.2 and 4.3 we did not commit ourselves as to the *cause* of **P**; we dealt only with the *effects* of polarization. From the qualitative discussion of Sect. 4.1, though, we know that the polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles. For many substances, in fact, the polarization is *proportional* to the field, provided **E** is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \tag{4.30}$$

The constant of proportionality, χ_e, is called the **electric susceptibility** of the medium (a factor of ϵ_0 has been extracted to make χ_e dimensionless). The value of χ_e depends on the microscopic structure of the substance in question (and also on external conditions such as temperature). I shall call materials that obey Eq. 4.30 **linear dielectrics.**[7]

Note that $\mathbf{E}$ in Eq. 4.30 is the *total* field; it may be due in part to free charges and in part to the polarization itself. If, for instance, we put a piece of dielectric into an external field $\mathbf{E}_0$, we cannot compute $\mathbf{P}$ directly from Eq. 4.30; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization, which . . . Breaking out of this infinite regress is not always easy. You'll see some examples in a moment. The simplest approach is to begin with the *displacement,* at least in those cases where $\mathbf{D}$ can be deduced directly from the free charge distribution.

In linear media we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E}, \tag{4.31}$$

so $\mathbf{D}$ is *also* proportional to $\mathbf{E}$:

$$\mathbf{D} = \epsilon\mathbf{E}, \tag{4.32}$$

where

$$\epsilon \equiv \epsilon_0(1 + \chi_e). \tag{4.33}$$

This new constant ϵ is called the **permittivity** of the material. (In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity is ϵ_0. That's why ϵ_0 is called the **permittivity of free space**. I dislike the term, for it suggests that the vacuum is just a special kind of linear dielectric, in which the permittivity happens to have the value $8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$.) If you remove a factor of ϵ_0, the remaining dimensionless quantity

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{4.34}$$

is called the **relative permittivity**, or **dielectric constant**, of the material. Dielectric constants for some common substances are listed in Table 4.2. (Notice that ϵ_r is greater than 1, for all ordinary materials.) Of course, the permittivity and the dielectric constant do not convey any information that was not already available in the susceptibility, nor is there anything essentially new in Eq. 4.32; the *physics* of linear dielectrics is all contained in Eq. 4.30.[8]

[7]In modern optical applications, especially, *non*linear materials have become increasingly important. For these there is a second term in the formula for $\mathbf{P}$ as a function of $\mathbf{E}$—typically a *cubic* term. In general, Eq. 4.30 can be regarded as the first (nonzero) term in the Taylor expansion of $\mathbf{P}$ in powers of $\mathbf{E}$.

[8]As long as we are engaged in this orgy of unnecessary terminology and notation, I might as well mention that formulas for $\mathbf{D}$ in terms of $\mathbf{E}$ (Eq. 4.32, in the case of linear dielectrics) are called **constitutive relations**.

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

TABLE 4.2 Dielectric Constants (unless otherwise specified, values given are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics,* 91st ed. (Boca Raton: CRC Press, 2010).

Example 4.5. A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution

To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using Eq. 4.32:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

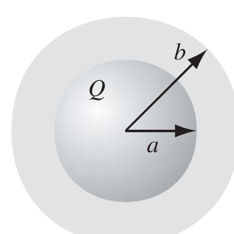

FIGURE 4.20

The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_{b}^{a}\left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_{a}^{0}(0)\, dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right).$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla\cdot\mathbf{P} = 0,$$

while

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\[2ex] \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$. In this respect, a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b$; the dielectric does the best it can, but the cancellation is only partial.

You might suppose that linear dielectrics escape the defect in the parallel between $\mathbf{E}$ and $\mathbf{D}$. Since $\mathbf{P}$ and $\mathbf{D}$ are now proportional to $\mathbf{E}$, does it not follow that their curls, like $\mathbf{E}$'s, must vanish? Unfortunately, it does *not,* for the line integral of $\mathbf{P}$ around a closed path that *straddles the boundary between one type of material and another* need not be zero, even though the integral of $\mathbf{E}$ around the same loop *must* be. The reason is that the proportionality factor $\epsilon_0\chi_e$ is different on the two sides. For instance, at the interface between a polarized dielectric and the vacuum (Fig. 4.21), $\mathbf{P}$ is zero on one side but not on the other. Around this

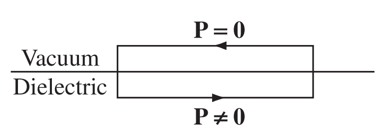

FIGURE 4.21

loop $\oint \mathbf{P} \cdot d\mathbf{l} \neq 0$, and hence, by Stokes' theorem, the curl of $\mathbf{P}$ cannot vanish everywhere within the loop (in fact, it is *infinite* at the boundary).[9]

Of course, if the space is *entirely* filled with a homogeneous[10] linear dielectric, then this objection is void; in this rather special circumstance

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = \mathbf{0},$$

so $\mathbf{D}$ can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}},$$

where $\mathbf{E}_{\text{vac}}$ is the field the same free charge distribution would produce in the absence of any dielectric. According to Eqs. 4.32 and 4.34, therefore,

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}. \tag{4.35}$$

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant. (Actually, it is not necessary for the dielectric to fill *all* space: in regions where the field is zero anyway, it can hardly matter whether the dielectric is present or not, since there's no polarization in any event.)

For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}} \tag{4.36}$$

(that's ϵ, not ϵ_0), and the force it exerts on nearby charges is reduced accordingly. But it's not that there is anything wrong with Coulomb's law; rather, the polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign (Fig. 4.22).[11]

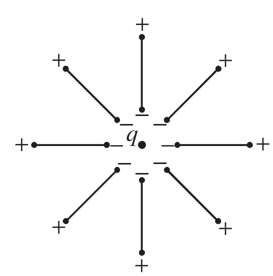

FIGURE 4.22

[9]Putting that argument in differential form, Eq. 4.30 and product rule 7 yield $\nabla \times \mathbf{P} = -\epsilon_0 \mathbf{E} \times (\nabla \chi_e)$, so the problem arises when $\nabla \chi_e$ is not parallel to $\mathbf{E}$.

[10]A **homogeneous** medium is one whose properties (in this case the susceptibility) do not vary with position.

[11]In *quantum* electrodynamics, the vacuum itself can be polarized, and this means that the effective (or "renormalized") charge of the electron, as you might measure it in the laboratory, is not its true ("bare") value, and in fact depends slightly on how far away you are!

Example 4.6. A parallel-plate capacitor (Fig. 4.23) is filled with insulating material of dielectric constant ϵ_r. What effect does this have on its capacitance?

Solution

Since the field is confined to the space between the plates, the dielectric will reduce $\mathbf{E}$, and hence also the potential difference V, by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is *increased by a factor of the dielectric constant,*

$$C = \epsilon_r C_{\text{vac}}. \tag{4.37}$$

This is, in fact, a common way to beef up a capacitor.

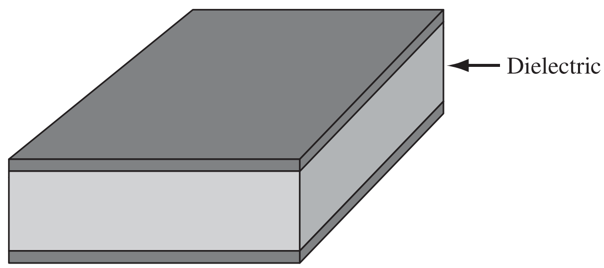

FIGURE 4.23

A *crystal* is generally easier to polarize in some directions than in others,[12] and in this case Eq. 4.30 is replaced by the general linear relation

$$\left.\begin{aligned} P_x &= \epsilon_0(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z) \\ P_y &= \epsilon_0(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z) \\ P_z &= \epsilon_0(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z) \end{aligned}\right\}, \tag{4.38}$$

just as Eq. 4.1 was superseded by Eq. 4.3 for asymmetrical molecules. The nine coefficients, $\chi_{e_{xx}}, \chi_{e_{xy}}, \ldots$, constitute the **susceptibility tensor.**

Problem 4.18 The space between the plates of a parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.

[12]A medium is said to be **isotropic** if its properties (such as susceptibility) are the same in all directions. Thus Eq. 4.30 is the special case of Eq. 4.38 that holds for isotropic media. Physicists tend to be sloppy with their language, and unless otherwise indicated the term "linear dielectric" implies "isotropic linear dielectric," and suggests "homogeneous isotropic linear dielectric." But technically, "linear" just means that at *any given point*, and for $\mathbf{E}$ in a *given direction*, the components of $\mathbf{P}$ are proportional to E—the proportionality factor could vary with position and/or direction.

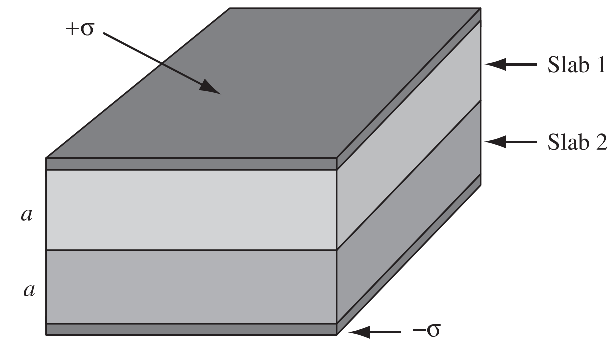

FIGURE 4.24

(a) Find the electric displacement **D** in each slab.

(b) Find the electric field **E** in each slab.

(c) Find the polarization **P** in each slab.

(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

Problem 4.19 Suppose you have enough linear dielectric material, of dielectric constant ϵ_r, to *half*-fill a parallel-plate capacitor (Fig. 4.25). By what fraction is the capacitance increased when you distribute the material as in Fig. 4.25(a)? How about Fig. 4.25(b)? For a given potential difference V between the plates, find **E**, **D**, and **P**, in each region, and the free and bound charge on all surfaces, for both cases.

Problem 4.20 A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and the dielectric constant is ϵ_r.

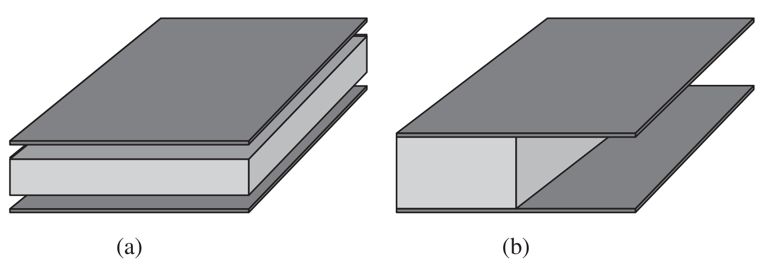

FIGURE 4.25

Problem 4.21 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable.

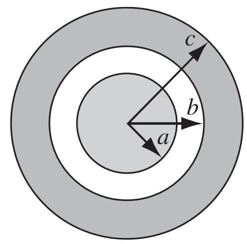

FIGURE 4.26

4.4.2 ■ Boundary Value Problems with Linear Dielectrics

In a (homogeneous isotropic) linear dielectric, the bound charge density (ρ_b) is proportional to the free charge density (ρ_f):[13]

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f. \tag{4.39}$$

In particular, unless free charge is actually embedded in the material, $\rho = 0$, and any net charge must reside at the surface. Within such a dielectric, then, the potential obeys Laplace's equation, and all the machinery of Chapter 3 carries over. It is convenient, however, to rewrite the boundary conditions in a way that makes reference only to the free charge. Equation 4.26 says

$$\epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f, \tag{4.40}$$

or (in terms of the potential),

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f, \tag{4.41}$$

whereas the potential itself is, of course, continuous (Eq. 2.34):

$$V_{\text{above}} = V_{\text{below}}. \tag{4.42}$$

[13]This does not apply to the surface charge (σ_b), because χ_e is not independent of position (obviously) at the boundary.

Example 4.7. A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$ (Fig. 4.27). Find the electric field inside the sphere.

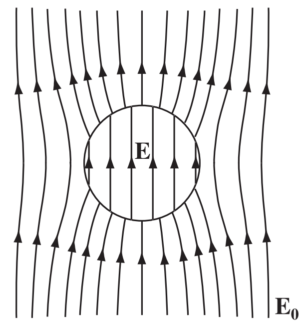

FIGURE 4.27

Solution

This is reminiscent of Ex. 3.8, in which an uncharged *conducting* sphere was introduced into a uniform field. In that case, the field of the induced charge canceled $\mathbf{E}_0$ within the sphere; in a *dielectric,* the cancellation (from the bound charge) is incomplete.

Our problem is to solve Laplace's equation, for $V_{\text{in}}(r, \theta)$ when $r \leq R$, and $V_{\text{out}}(r, \theta)$ when $r \geq R$, subject to the boundary conditions

$$\left.\begin{array}{llll} \text{(i)} & V_{\text{in}} = V_{\text{out}}, & & \text{at } r = R, \\ \text{(ii)} & \epsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r}, & & \text{at } r = R, \\ \text{(iii)} & V_{\text{out}} \rightarrow -E_0 r \cos\theta, & & \text{for } r \gg R. \end{array}\right\} \tag{4.43}$$

(The second of these follows from Eq. 4.41, since there is no free charge at the surface.) Inside the sphere, Eq. 3.65 says

$$V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} A_l \, r^l P_l(\cos\theta); \tag{4.44}$$

outside the sphere, in view of (iii), we have

$$V_{\text{out}}(r, \theta) = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{4.45}$$

Boundary condition (i) requires that

$$\sum_{l=0}^{\infty} A_l \, R^l P_l(\cos\theta) = -E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta),$$

so[14]

$$\left.\begin{aligned} A_l R^l &= \frac{B_l}{R^{l+1}}, \quad \text{for } l \neq 1, \\ A_1 R &= -E_0 R + \frac{B_1}{R^2}. \end{aligned}\right\} \tag{4.46}$$

Meanwhile, condition (ii) yields

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta),$$

so

$$\left.\begin{aligned} \epsilon_r l A_l R^{l-1} &= -\frac{(l+1)B_l}{R^{l+2}}, \text{ for } l \neq 1, \\ \epsilon_r A_1 &= -E_0 - \frac{2B_1}{R^3}. \end{aligned}\right\} \tag{4.47}$$

It follows that

$$\left.\begin{aligned} &A_l = B_l = 0, && \text{for } l \neq 1, \\ &A_1 = -\frac{3}{\epsilon_r + 2} E_0 && B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0. \end{aligned}\right\} \tag{4.48}$$

Evidently

$$V_{\text{in}}(r,\theta) = -\frac{3E_0}{\epsilon_r + 2} r\cos\theta = -\frac{3E_0}{\epsilon_r + 2} z,$$

and hence the field inside the sphere is (surprisingly) *uniform*:

$$\mathbf{E} = \frac{3}{\epsilon_r + 2}\mathbf{E}_0. \tag{4.49}$$

Example 4.8. Suppose the entire region below the plane $z = 0$ in Fig. 4.28 is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated a distance d above the origin.

[14]Remember, $P_1(\cos\theta) = \cos\theta$, and the coefficients must be equal for each l, as you could prove by multiplying by $P_{l'}(\cos\theta)\sin\theta$, integrating from 0 to π, and invoking the orthogonality of the Legendre polynomials (Eq. 3.68).

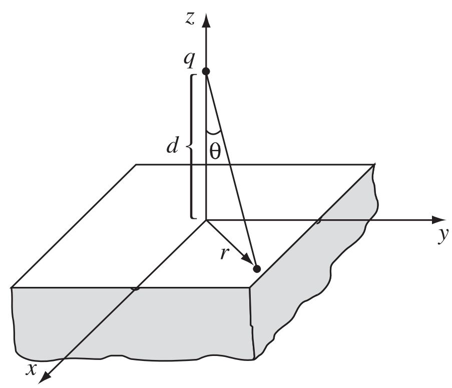

FIGURE 4.28

Solution
The surface bound charge on the xy plane is of opposite sign to q, so the force will be attractive. (In view of Eq. 4.39, there is no volume bound charge.) Let us first calculate σ_b, using Eqs. 4.11 and 4.30.[15]

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P_z = \epsilon_0 \chi_e E_z,$$

where E_z is the z-component of the total field just inside the dielectric, at $z = 0$. This field is due in part to q and in part to the bound charge itself. From Coulomb's law, the former contribution is

$$-\frac{1}{4\pi\epsilon_0} \frac{q}{(r^2 + d^2)} \cos\theta = -\frac{1}{4\pi\epsilon_0} \frac{qd}{(r^2 + d^2)^{3/2}},$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the origin. The z component of the field of the bound charge, meanwhile, is $-\sigma_b/2\epsilon_0$ (see footnote after Eq. 2.33). Thus

$$\sigma_b = \epsilon_0 \chi_e \left[-\frac{1}{4\pi\epsilon_0} \frac{qd}{(r^2 + d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0} \right],$$

which we can solve for σ_b:

$$\sigma_b = -\frac{1}{2\pi} \left(\frac{\chi_e}{\chi_e + 2} \right) \frac{qd}{(r^2 + d^2)^{3/2}}. \tag{4.50}$$

Apart from the factor $\chi_e/(\chi_e + 2)$, this is exactly the same as the induced charge on an infinite *conducting* plane under similar circumstances (Eq. 3.10).[16] Evidently the *total* bound charge is

$$q_b = -\left(\frac{\chi_e}{\chi_e + 2} \right) q. \tag{4.51}$$

[15]This method mimics Prob. 3.38.

[16]For some purposes a conductor can be regarded as the limiting case of a linear dielectric, with $\chi_e \to \infty$. This is often a useful check—try applying it to Exs. 4.5, 4.6, and 4.7.

We could, of course, obtain the field of σ_b by direct integration

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\sigma_b\,da.$$

But, as in the case of the conducting plane, there is a nicer solution by the method of images. Indeed, if we replace the dielectric by a single point charge q_b at the image position $(0, 0, -d)$, we have

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right], \tag{4.52}$$

in the region $z > 0$. Meanwhile, a charge $(q + q_b)$ at $(0, 0, d)$ yields the potential

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q+q_b}{\sqrt{x^2+y^2+(z-d)^2}}\right], \tag{4.53}$$

for the region $z < 0$. Taken together, Eqs. 4.52 and 4.53 constitute a function that satisfies Poisson's equation with a point charge q at $(0, 0, d)$, which goes to zero at infinity, which is continuous at the boundary $z = 0$, and whose normal derivative exhibits the discontinuity appropriate to a surface charge σ_b at $z = 0$:

$$-\epsilon_0\left(\left.\frac{\partial V}{\partial z}\right|_{z=0^+} - \left.\frac{\partial V}{\partial z}\right|_{z=0^-}\right) = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(x^2+y^2+d^2)^{3/2}}.$$

Accordingly, this is the correct potential for our problem. In particular, the force on q is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}\hat{\mathbf{z}}. \tag{4.54}$$

I do not claim to have provided a compelling *motivation* for Eqs. 4.52 and 4.53—like all image solutions, this one owes its justification to the fact that it *works:* it solves Poisson's equation, and it meets the boundary conditions. Still, discovering an image solution is not entirely a matter of guesswork. There are at least two "rules of the game": (1) You must never put an image charge into the region where you're computing the potential. (Thus Eq. 4.52 gives the potential for $z > 0$, but this image charge q_b is at $z = -d$; when we turn to the region $z < 0$ (Eq. 4.53), the image charge $(q + q_b)$ is at $z = +d$.) (2) The image charges must add up to the correct total in each region. (That's how I knew to use q_b to account for the charge in the region $z \le 0$, and $(q + q_b)$ to cover the region $z \ge 0$.)

Problem 4.22 A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting field within the cylinder. (The radius is a, the susceptibility χ_e, and the axis is perpendicular to $\mathbf{E}_0$.)

Problem 4.23 Find the field inside a sphere of linear dielectric material in an otherwise uniform electric field $\mathbf{E}_0$ (Ex. 4.7) by the following method of successive approximations: First pretend the field inside is just $\mathbf{E}_0$, and use Eq. 4.30 to write down the resulting polarization $\mathbf{P}_0$. This polarization generates a field of its own, $\mathbf{E}_1$ (Ex. 4.2), which in turn modifies the polarization by an amount $\mathbf{P}_1$, which further changes the field by an amount $\mathbf{E}_2$, and so on. The resulting field is $\mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots$. Sum the series, and compare your answer with Eq. 4.49.

Problem 4.24 An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b. This object is now placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the electric field in the insulator.

! **Problem 4.25** Suppose the region *above* the xy plane in Ex. 4.8 is *also* filled with linear dielectric but of a different susceptibility χ_e'. Find the potential everywhere.

4.4.3 ■ Energy in Dielectric Systems

It takes work to charge up a capacitor (Eq. 2.55):

$$W = \tfrac{1}{2}CV^2.$$

If the capacitor is filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of the dielectric constant,

$$C = \epsilon_r C_{\text{vac}},$$

as we found in Ex. 4.6. Evidently the work necessary to charge a dielectric-filled capacitor is increased by the same factor. The reason is pretty clear: you have to pump on more (free) charge, to achieve a given potential, because part of the field is canceled off by the bound charges.

In Chapter 2, I derived a general formula for the energy stored in any electrostatic system (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\,d\tau. \tag{4.55}$$

The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\,d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau,$$

in the presence of linear dielectrics. To *prove* it, suppose the dielectric material is fixed in position, and we bring in the free charge, a bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will change and with it the bound charge distribution; but we're interested only in the work done on the incremental *free* charge:

$$\Delta W = \int (\Delta\rho_f)V\,d\tau. \tag{4.56}$$

Since $\nabla \cdot \mathbf{D} = \rho_f$, $\Delta\rho_f = \nabla \cdot (\Delta\mathbf{D})$, where $\Delta\mathbf{D}$ is the resulting change in $\mathbf{D}$, so

$$\Delta W = \int [\nabla \cdot (\Delta\mathbf{D})] V \, d\tau.$$

Now

$$\nabla \cdot [(\Delta\mathbf{D})V] = [\nabla \cdot (\Delta\mathbf{D})]V + \Delta\mathbf{D} \cdot (\nabla V),$$

and hence (integrating by parts):

$$\Delta W = \int \nabla \cdot [(\Delta\mathbf{D})V] \, d\tau + \int (\Delta\mathbf{D}) \cdot \mathbf{E} \, d\tau.$$

The divergence theorem turns the first term into a surface integral, which vanishes if we integrate over all space. Therefore, the work done is equal to

$$\Delta W = \int (\Delta\mathbf{D}) \cdot \mathbf{E} \, d\tau. \tag{4.57}$$

So far, this applies to *any* material. Now, if the medium is a linear dielectric, then $\mathbf{D} = \epsilon\mathbf{E}$, so

$$\tfrac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E}) = \tfrac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\mathbf{E}) \cdot \mathbf{E} = (\Delta\mathbf{D}) \cdot \mathbf{E}$$

(for infinitesimal increments). Thus

$$\Delta W = \Delta\left(\frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} \, d\tau\right).$$

The total work done, then, as we build the free charge up from zero to the final configuration, is

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} \, d\tau, \tag{4.58}$$

as anticipated.[17]

It may puzzle you that Eq. 4.55, which we derived quite generally in Chapter 2, does not seem to apply in the presence of dielectrics, where it is replaced by Eq. 4.58. The point is not that one or the other of these equations is *wrong,* but rather that they address somewhat different questions. The distinction is subtle, so let's go right back to the beginning: What do we *mean* by "the energy of a system"? *Answer:* It is the work required to assemble the system. Very

[17] In case you are wondering why I did not do this more simply by the method of Sect. 2.4.3, starting with $W = \frac{1}{2}\int \rho_f V \, d\tau$, the reason is that *this* formula is untrue, in general. Study the derivation of Eq. 2.42, and you will see that it applies only to the *total* charge. For *linear* dielectrics it happens to hold for the free charge alone, but this is scarcely obvious a priori and, in fact, is most easily confirmed by working backward from Eq. 4.58.

well—but when dielectrics are involved, there are two quite different ways one might construe this process:

1. We bring in all the charges (free *and* bound), one by one, with tweezers, and glue each one down in its proper final location. If *this* is what you mean by "assemble the system," then Eq. 4.55 is your formula for the energy stored. Notice, however, that this will *not* include the work involved in stretching and twisting the dielectric molecules (if we picture the positive and negative charges as held together by tiny springs, it does not include the spring energy, $\frac{1}{2}kx^2$, associated with polarizing each molecule).[18]

2. With the unpolarized dielectric in place, we bring in the *free* charges, one by one, allowing the dielectric to respond as it sees fit. If *this* is what you mean by "assemble the system" (and ordinarily it *is,* since free charge is what we actually push around), then Eq. 4.58 is the formula you want. In this case the "spring" energy *is* included, albeit indirectly, because the force you must apply to the *free* charge depends on the disposition of the *bound* charge; as you move the free charge, you are automatically stretching those "springs."

Example 4.9. A sphere of radius R is filled with material of dielectric constant ϵ_r and uniform embedded free charge ρ_f. What is the energy of this configuration?

Solution

From Gauss's law (in the form of Eq. 4.23), the displacement is

$$\mathbf{D}(r) = \begin{cases} \dfrac{\rho_f}{3}\mathbf{r} & (r < R), \\ \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

So the electric field is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < R), \\ \dfrac{\rho_f}{3\epsilon_0}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The purely *electrostatic* energy (Eq. 4.55) is

$$\begin{aligned} W_1 &= \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)^2 \int_0^R r^2\, 4\pi r^2\, dr + \left(\frac{\rho_f}{3\epsilon_0}\right)^2 R^6 \int_R^\infty \frac{1}{r^4}\, 4\pi r^2\, dr\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r^2} + 1\right). \end{aligned}$$

[18] The "spring" itself may be electrical in nature, but it is still not included in Eq. 4.55, if **E** is taken to be the *macroscopic* field.

But the *total* energy (Eq. 4.58) is

$$W_2 = \frac{1}{2}\left[\left(\frac{\rho_f}{3}\right)\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)\int_0^R r^2\,4\pi r^2\,dr + \left(\frac{\rho_f R^3}{3}\right)\left(\frac{\rho_f R^3}{3\epsilon_0}\right)\int_R^\infty \frac{1}{r^4}\,4\pi r^2\,dr\right]$$
$$= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r}+1\right).$$

Notice that $W_1 < W_2$—that's because W_1 does not include the energy involved in stretching the molecules.

Let's check that W_2 is the work done on the *free* charge in assembling the system. We start with the (uncharged, unpolarized) dielectric sphere, and bring in the free charge in infinitesimal installments (dq), filling out the sphere layer by layer. When we have reached radius r', the electric field is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < r'), \\[2ex] \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r' < r < R), \\[2ex] \dfrac{\rho_f}{3\epsilon_0}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The work required to bring the next dq in from infinity to r' is

$$dW = -dq\left[\int_\infty^R \mathbf{E}\cdot d\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot d\mathbf{l}\right]$$
$$= -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}\,dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'} \frac{1}{r^2}\,dr\right]$$
$$= \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right]dq.$$

This increases the radius (r'):

$$dq = \rho_f 4\pi r'^2\,dr',$$

so the *total* work done, in going from $r' = 0$ to $r' = R$, is

$$W = \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1 - \frac{1}{\epsilon_r}\right)\int_0^R r'^5\,dr' + \frac{1}{\epsilon_r}\int_0^R r'^4\,dr'\right]$$
$$= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r}+1\right) = W_2. \checkmark$$

Evidently the energy "stored in the springs" is

$$W_{\text{spring}} = W_2 - W_1 = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5\,(\epsilon_r - 1).$$

I would like to confirm this in an explicit model. Picture the dielectric as a collection of tiny proto-dipoles, each consisting of $+q$ and $-q$ attached to a spring of constant k and equilibrium length 0, so in the absence of any field the positive and negative ends coincide. One end of each dipole is nailed in position (like the nuclei in a solid), but the other end is free to move in response to any imposed field. Let $d\tau$ be the volume assigned to each proto-dipole (the dipole itself may occupy only a small portion of this space).

With the field turned on, the electric force on the free end is balanced by the spring force;[19] the charges separate by a distance d: $qE = kd$. In our case

$$\mathbf{E} = \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r}.$$

The resulting dipole moment is $p = qd$, and the polarization is $P = p/d\tau$, so

$$k = \frac{\rho_f}{3\epsilon_0\epsilon_r d^2} Pr\, d\tau.$$

The energy of this particular spring is

$$dW_{\text{spring}} = \frac{1}{2}kd^2 = \frac{\rho_f}{6\epsilon_0\epsilon_r} Pr\, d\tau,$$

and hence the total is

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\int Pr\, d\tau.$$

Now

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \epsilon_0\chi_e\frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} = \frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}\mathbf{r},$$

so

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}4\pi\int_0^R r^4\, dr = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5\,(\epsilon_r - 1)\,,$$

and it works out perfectly.

It is sometimes alleged that Eq. 4.58 represents the energy even for *non*linear dielectrics, but this is false: To proceed beyond Eq. 4.57, one must assume linearity. In fact, for *dissipative* systems the whole notion of "stored energy" loses its meaning, because the work done depends not only on the final configuration but on *how it got there.* If the molecular "springs" are allowed to have some

[19]Note that the "spring" here is a surrogate for whatever holds the molecule together—it *includes* the electrical attraction of the other end. If it bothers you that the force is taken to be proportional to the separation, look again at Example 4.1.

friction, for instance, then W_{spring} can be made as large as you like, by assembling the charges in such a way that the spring is obliged to expand and contract many times before reaching its final state. In particular, you get nonsensical results if you try to apply Eq. 4.58 to electrets, with frozen-in polarization (see Prob. 4.27).

Problem 4.26 A spherical conductor, of radius a, carries a charge Q (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration (Eq. 4.58).

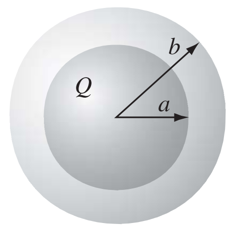

FIGURE 4.29

Problem 4.27 Calculate W, using both Eq. 4.55 and Eq. 4.58, for a sphere of radius R with frozen-in uniform polarization $\mathbf{P}$ (Ex. 4.2). Comment on the discrepancy. Which (if either) is the "true" energy of the system?

4.4.4 ■ Forces on Dielectrics

Just as a conductor is attracted into an electric field (Eq. 2.51), so too is a dielectric—and for essentially the same reason: the bound charge tends to accumulate near the free charge of the opposite sign. But the calculation of forces on dielectrics can be surprisingly tricky. Consider, for example, the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor (Fig. 4.30). We have always pretended that the field is uniform inside a parallel-plate capacitor, and zero outside. If this were literally true, there would be no net force on the dielectric at all, since the field everywhere would be perpendicular to the plates. However, there is in reality a **fringing field** around the edges, which for most purposes can be ignored but in this case is responsible for the whole effect. (Indeed, the field *could* not terminate abruptly at the edge of the capacitor, for if it did, the line integral of $\mathbf{E}$ around the closed loop shown in Fig. 4.31 would not be zero.) It is this nonuniform fringing field that pulls the dielectric into the capacitor.

Fringing fields are notoriously difficult to calculate; luckily, we can avoid this altogether, by the following ingenious method.[20] Let W be the energy of the

[20] For a direct calculation from the fringing fields, see E. R. Dietz, *Am. J. Phys.* **72**, 1499 (2004).

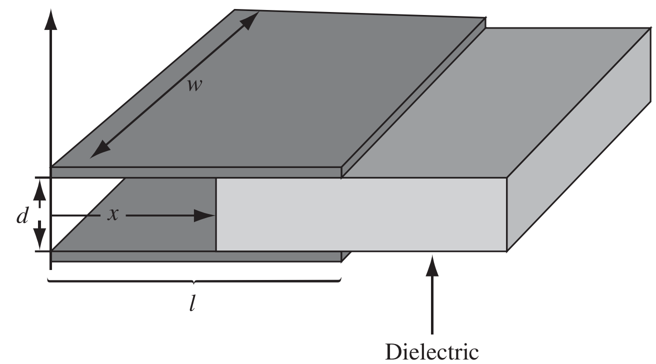

FIGURE 4.30

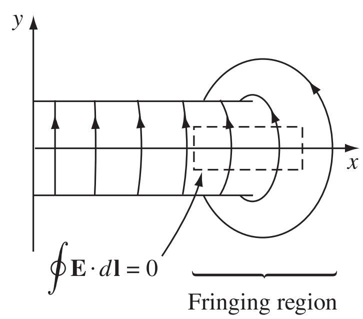

FIGURE 4.31

system—it depends, of course, on the amount of overlap. If I pull the dielectric out an infinitesimal distance dx, the energy is changed by an amount equal to the work done:

$$dW = F_{\text{me}}\, dx, \tag{4.59}$$

where F_{me} is the force I must exert, to counteract the electrical force F on the dielectric: $F_{\text{me}} = -F$. Thus the electrical force on the slab is

$$F = -\frac{dW}{dx}. \tag{4.60}$$

Now, the energy stored in the capacitor is

$$W = \tfrac{1}{2}CV^2, \tag{4.61}$$

and the capacitance in this case is

$$C = \frac{\epsilon_0 w}{d}(\epsilon_r l - \chi_e x), \tag{4.62}$$

where l is the length of the plates (Fig. 4.30). Let's assume that the total charge on the plates ($Q = CV$) is held constant, as the dielectric moves. In terms of Q,

$$W = \frac{1}{2}\frac{Q^2}{C}, \tag{4.63}$$

so

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}. \tag{4.64}$$

But

$$\frac{dC}{dx} = -\frac{\epsilon_0\chi_e w}{d},$$

and hence

$$F = -\frac{\epsilon_0\chi_e w}{2d}V^2. \tag{4.65}$$

(The minus sign indicates that the force is in the negative x direction; the dielectric is pulled *into* the capacitor.)

It is a common error to use Eq. 4.61 (with V constant), rather than Eq. 4.63 (with Q constant), in computing the force. One then obtains

$$F = -\frac{1}{2}V^2\frac{dC}{dx},$$

which is off by a sign. It is, of course, *possible* to maintain the capacitor at a fixed potential, by connecting it up to a battery. But in that case the *battery also does work* as the dielectric moves; instead of Eq. 4.59, we now have

$$dW = F_{\text{me}}\,dx + V\,dQ, \tag{4.66}$$

where $V\,dQ$ is the work done by the battery. It follows that

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}, \tag{4.67}$$

the same as before (Eq. 4.64), with the *correct* sign.

Please understand: The force on the dielectric cannot possibly depend on whether you plan to hold Q constant or V constant—it is determined entirely by the distribution of charge, free and bound. It's simpler to *calculate* the force assuming constant Q, because then you don't have to worry about work done by the battery; but if you insist, it can be done correctly either way.

Notice that we were able to determine the force *without knowing anything about the fringing fields that are ultimately responsible for it!* Of course, it's built into the whole structure of electrostatics that $\nabla \times \mathbf{E} = \mathbf{0}$, and hence that the fringing fields must be present; we're not really getting something for nothing here—just cleverly exploiting the internal consistency of the theory. The energy stored in the fringing fields themselves (which was not accounted for in this derivation) stays constant, as the slab moves; what *does* change is the energy well *inside* the capacitor, where the field is nice and uniform.

Problem 4.28 Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded (Fig. 4.32). To what height (h) does the oil rise, in the space between the tubes?

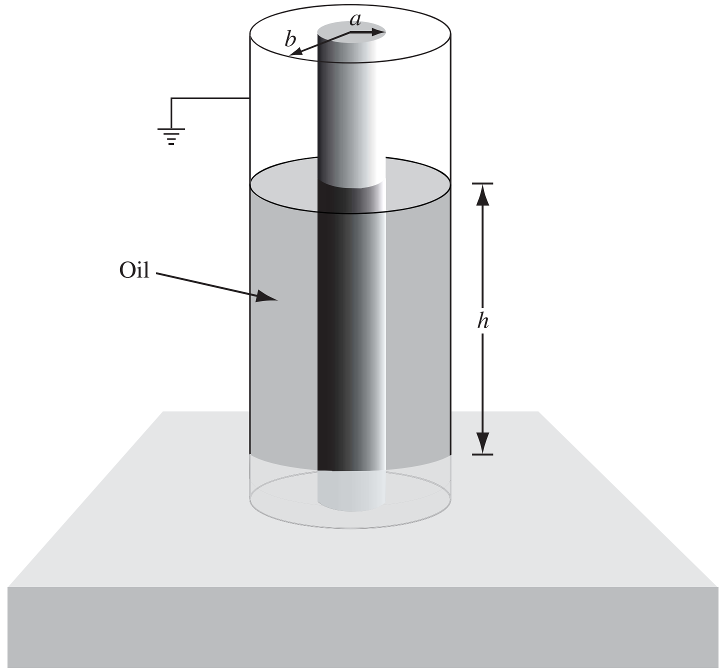

FIGURE 4.32

More Problems on Chapter 4

Problem 4.29

(a) For the configuration in Prob. 4.5, calculate the *force* on $\mathbf{p}_2$ due to $\mathbf{p}_1$, and the force on $\mathbf{p}_1$ due to $\mathbf{p}_2$. Are the answers consistent with Newton's third law?

(b) Find the total torque on $\mathbf{p}_2$ *with respect to the center of* $\mathbf{p}_1$, and compare it with the torque on $\mathbf{p}_1$ *about that same point*. [*Hint:* combine your answer to (a) with the result of Prob. 4.5.]

Problem 4.30 An electric dipole $\mathbf{p}$, pointing in the y direction, is placed midway between two large conducting plates, as shown in Fig. 4.33. Each plate makes a

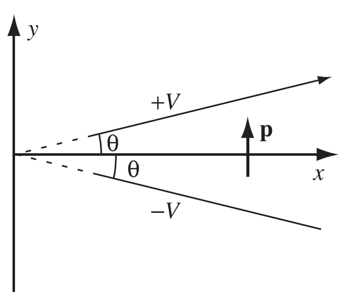

FIGURE 4.33

small angle θ with respect to the x axis, and they are maintained at potentials $\pm V$. What is the *direction* of the net force on $\mathbf{p}$? (There's nothing to *calculate*, here, but do explain your answer qualitatively.)

Problem 4.31 A point charge Q is "nailed down" on a table. Around it, at radius R, is a frictionless circular track on which a dipole $\mathbf{p}$ rides, constrained always to point tangent to the circle. Use Eq. 4.5 to show that the electric force on the dipole is

$$\mathbf{F} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}.$$

Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]

! **Problem 4.32** Earnshaw's theorem (Prob. 3.2) says that you cannot trap a charged particle in an electrostatic field. *Question:* Could you trap a neutral (but polarizable) atom in an electrostatic field?

(a) Show that the force on the atom is $\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2)$.

(b) The question becomes, therefore: Is it possible for E^2 to have a local maximum (in a charge-free region)? In that case the force would push the atom back to its equilibrium position. Show that the answer is *no*. [*Hint:* Use Prob. 3.4(a).][22]

Problem 4.33 A dielectric cube of side a, centered at the origin, carries a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.

Problem 4.34 The space between the plates of a parallel-plate capacitor is filled with dielectric material whose dielectric constant varies linearly from 1 at the bottom plate ($x = 0$) to 2 at the top plate ($x = d$). The capacitor is connected to a battery of voltage V. Find all the bound charge, and check that the total is zero.

Problem 4.35 A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

Problem 4.36 At the interface between one linear dielectric and another, the electric field lines bend (see Fig. 4.34). Show that

$$\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_1, \tag{4.68}$$

assuming there is no *free* charge at the boundary. [*Comment:* Eq. 4.68 is reminiscent of Snell's law in optics. Would a convex "lens" of dielectric material tend to "focus," or "defocus," the electric field?]

[21]This charming paradox was suggested by K. Brownstein.

[22]Interestingly, it *can* be done with *oscillating* fields. See K. T. McDonald, *Am. J. Phys.* **68**, 486 (2000).

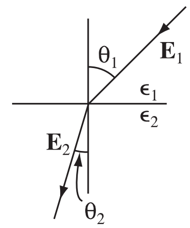

FIGURE 4.34

! **Problem 4.37** A point dipole $\mathbf{p}$ is imbedded at the center of a sphere of linear dielectric material (with radius R and dielectric constant ϵ_r). Find the electric potential inside and outside the sphere.

$$\left[\textit{Answer: } \frac{p\cos\theta}{4\pi\epsilon r^2}\left(1+2\frac{r^3}{R^3}\frac{(\epsilon_r-1)}{(\epsilon_r+2)}\right),\ (r\le R);\ \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\left(\frac{3}{\epsilon_r+2}\right),\ (r\ge R)\right]$$

Problem 4.38 Prove the following uniqueness theorem: A volume $\mathcal{V}$ contains a specified free charge distribution, and various pieces of linear dielectric material, with the susceptibility of each one given. If the potential is specified on the boundaries $\mathcal{S}$ of $\mathcal{V}$ ($V = 0$ at infinity would be suitable) then the potential throughout $\mathcal{V}$ is uniquely determined. [*Hint:* Integrate $\boldsymbol{\nabla}\cdot(V_3\mathbf{D}_3)$ over $\mathcal{V}$.]

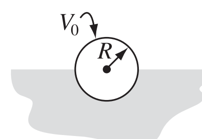

FIGURE 4.35

Problem 4.39 A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ (Fig. 4.35). *Claim:* the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check this claim, as follows:

(a) Write down the formula for the proposed potential $V(r)$, in terms of V_0, R, and r. Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.

(b) Show that the resulting charge configuration would indeed produce the potential $V(r)$.

(c) Appeal to the uniqueness theorem in Prob. 4.38 to complete the argument.

(d) Could you solve the configurations in Fig. 4.36 with the same potential? If not, explain *why*.

FIGURE 4.36

Problem 4.40 According to Eq. 4.5, the force on a single dipole is $(\mathbf{p}\cdot\nabla)\mathbf{E}$, so the *net* force on a dielectric object is

$$\mathbf{F} = \int (\mathbf{P}\cdot\nabla)\mathbf{E}_{\text{ext}}\, d\tau. \tag{4.69}$$

[Here $\mathbf{E}_{\text{ext}}$ is the field of everything *except* the dielectric. You might assume that it wouldn't matter if you used the *total* field; after all, the dielectric can't exert a force on *itself.* However, because the field of the dielectric is discontinuous at the location of any bound surface charge, the derivative introduces a spurious delta function, and it is safest to stick with $\mathbf{E}_{\text{ext}}$.] Use Eq. 4.69 to determine the force on a tiny sphere, of radius R, composed of linear dielectric material of susceptibility χ_e, which is situated a distance s from a fine wire carrying a uniform line charge λ.

! **Problem 4.41** In a linear dielectric, the polarization is proportional to the field: $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$. If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field $\mathbf{p} = \alpha\mathbf{E}$. *Question:* What is the relation between the atomic polarizability α and the susceptibility χ_e?

Since $\mathbf{P}$ (the dipole moment per unit volume) is $\mathbf{p}$ (the dipole moment per atom) times N (the number of atoms per unit volume), $\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}$, one's first inclination is to say that

$$\chi_e = \frac{N\alpha}{\epsilon_0}. \tag{4.70}$$

And in fact this is not far off, if the density is low. But closer inspection reveals a subtle problem, for the field $\mathbf{E}$ in Eq. 4.30 is the *total macroscopic* field in the medium, whereas the field in Eq. 4.1 is due to everything *except* the particular atom under consideration (polarizability was defined for an isolated atom subject to a specified external field); call this field $\mathbf{E}_{\text{else}}$. Imagine that the space allotted to each atom is a sphere of radius R, and show that

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}. \tag{4.71}$$

Use this to conclude that

$$\chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0},$$

or

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right). \tag{4.72}$$

rest of the atom. This makes it easy for the electron to leave the residual P^- ion and join the conduction band, as in Fig. 4.11(a).

This brings us to a more general problem. What if the space in our system is partly filled with dielectric and partly empty, with electric fields in both parts? We'll begin with a somewhat artificial but instructive example, a polarized solid sphere in otherwise empty space.

10.9 The field of a polarized sphere

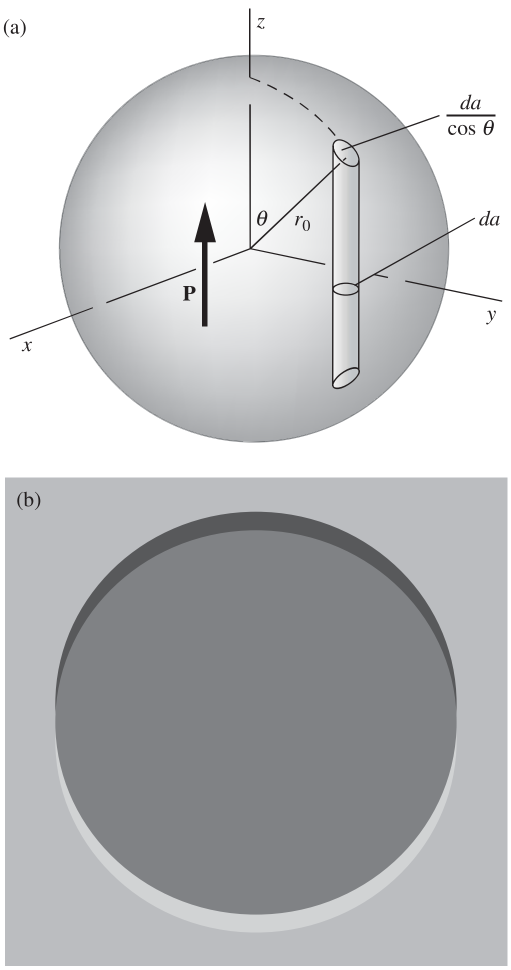

Figure 10.21.
(a) Divide the polarized sphere into polarized rods, and replace each rod by patches of charge on the surface of the sphere. (b) A ball of positive volume charge density and a ball of negative volume charge density, slightly displaced, are equivalent to a distribution of charge on the spherical surface.

The solid sphere in Fig. 10.21(a) is supposed to be uniformly polarized, as if it had been carved out of the substance of the slab in Fig. 10.16(a). What must the electric field be like, both inside and outside the sphere? We take **P** as usual to denote the density of polarization, constant in magnitude and direction throughout the volume of the sphere. The polarized material could be divided, like the slab in Fig. 10.16(a), into columns parallel to **P**, and each of these replaced by a charge of magnitude $P \times$ (column cross section) at top and bottom. Thus the field we seek is that of a surface charge distribution spread over a sphere with density $\sigma = P\cos\theta$. The factor $\cos\theta$ enters, as should be evident from the figure, because a column of cross section da intercepts on the sphere a patch of surface of area $da/\cos\theta$. Figure 10.21(b) is a cross section through this shell of equivalent surface charge in which the density of charge has been indicated by the varying thickness of the black semicircle above (positive charge density) and the light semicircle below (negative charge density).

If it has not already occurred to you, this figure may suggest that we think of the polarization **P** as having arisen from the slight upward displacement of a ball filled uniformly with positive charge of volume density ρ, relative to a ball of negative charge of density $-\rho$. That would leave uncompensated positive charge poking out at the top and negative charge showing at the bottom, varying in amount precisely as $\cos\theta$ over the whole boundary.[4] In the interior, where the positive and negative charge densities still overlap, they would exactly cancel one another. Taking this view, we see a very easy way to calculate the field *outside* the shell of surface charge. Any spherical charge distribution, as we know, has an external field the same as if its entire charge were concentrated at the center. So the superposition of two spheres of total charge Q and $-Q$, with their centers separated by a small displacement s, will produce an external field the same as that of two point charges Q and $-Q$, a distance s apart. This is just a dipole with dipole moment $p_0 = Qs$.

A microscopic description of the polarized substance leads us to the same conclusion. In Fig. 10.22(a) the molecular dipoles actually responsible for the polarization **P** have been crudely represented as consisting individually of a pair of charges q and $-q$, a distance s apart, to make

[4] This follows from the fact that the thickness of the "semicircle" at a given point is the radial component of the vertical vector representing the displacement s of the top sphere relative to the bottom sphere. You can quickly show that this radial component is $s\cos\theta$.

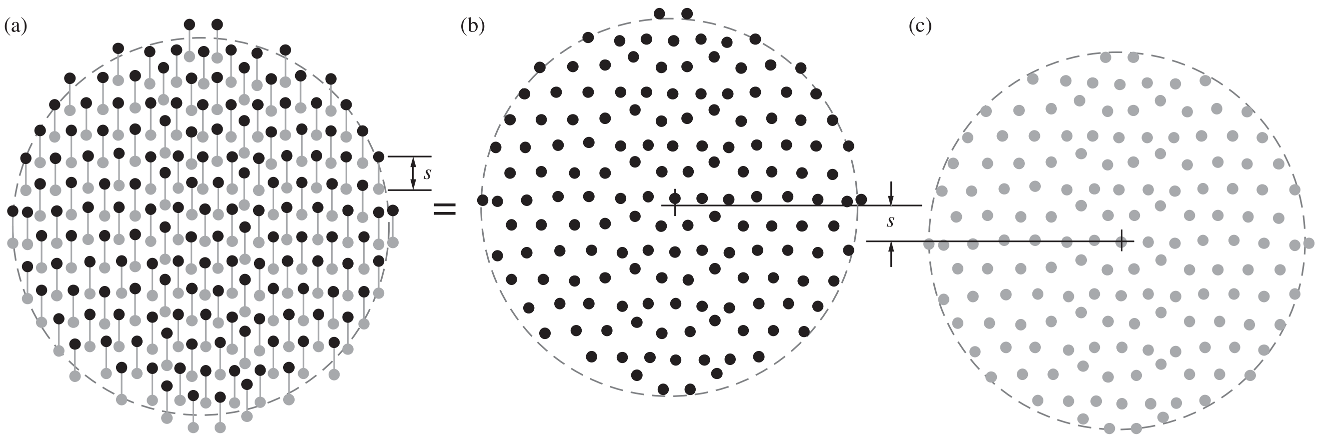

Figure 10.22.
A sphere of lined-up molecular dipoles (a) is equivalent to superposed, slightly displaced, spheres of positive (b) and negative (c) charges.

a dipole moment $p = qs$. With N of these per cubic meter, we have $P = Np = Nqs$, and the total number of such dipoles in the sphere is $(4\pi/3)r_0^3N$. The positive charges, considered separately (Fig. 10.22(b)), are distributed throughout a sphere with total charge content $Q = (4\pi/3)r_0^3Nq$, and the negative charges occupy a similar sphere with its center displaced (Fig. 10.22(c)). Clearly each of these charge distributions can be replaced by a point charge at its center, if we are concerned with the field well outside the distribution. "Well outside" means far enough away from the surface so that the actual graininess of the charge distribution doesn't matter, and of course that is something we always have to ignore when we speak of the macroscopic fields.

So, for present purposes, the picture of overlapping spheres of uniform charge density and the description in terms of actual dipoles in a vacuum are equivalent,[5] and show that the field outside the distribution is the same as that of a single dipole located at the center. The moment of this dipole p_0 is simply the total polarization in the sphere:

$$p_0 = Qs = \frac{4\pi}{3}r_0^3Nqs = \frac{4\pi}{3}r_0^3P. \tag{10.43}$$

The quantities Q and s have, separately, no significance and may now be dropped from the discussion.

The external field of the polarized sphere is that of a central dipole p_0, not only at a great distance from the sphere but also right down to the surface, macroscopically speaking. All we had to do to construct Fig. 10.23, a representation of the external field lines, was to block out a circular area from Fig. 10.6.

The internal field is a different matter. Let's look at the electric potential, $\phi(x, y, z)$. We know the potential at all points on the spherical

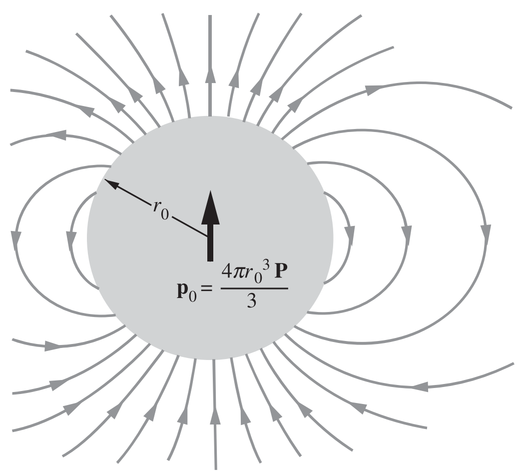

Figure 10.23.
The field outside a uniformly polarized sphere is exactly the same as that of a dipole located at the center of the sphere.

[5] This may have been obvious enough, but we have labored the details in this one case to allay any suspicion that the "smooth-charge-ball" picture, which is so different from what we know the interior of a real substance to be like, might be leading us astray.

boundary because we know the external field. It is just the dipole potential, $p_0 \cos\theta/4\pi\epsilon_0 r^2$, which on the spherical boundary of radius r_0 becomes

$$\phi = p_0 \frac{\cos\theta}{4\pi\epsilon_0 r_0^2} = \frac{Pr_0\cos\theta}{3\epsilon_0}, \qquad (10.44)$$

where we have used Eq. (10.43). Since $r_0 \cos\theta = z$, we see that the potential of a point on the sphere depends only on its z coordinate:

$$\phi = \frac{Pz}{3\epsilon_0}. \qquad (10.45)$$

The problem of finding the internal field has boiled down to this: Eq. (10.45) gives the potential at every point on the boundary of the region, inside which ϕ must satisfy Laplace's equation. According to the uniqueness theorem we proved in Chapter 3, that suffices to determine ϕ throughout the interior. If we can find *a* solution, it must be *the* solution. Now the function Cz, where C is any constant, satisfies Laplace's equation, so Eq. (10.45) has actually handed us the solution to the potential *in the interior* of the sphere. That is, $\phi_{\text{in}} = Pz/3\epsilon_0$. The electric field associated with this potential is uniform and points in the $-z$ direction:

$$E_z = -\frac{\partial\phi_{\text{in}}}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{Pz}{3\epsilon_0}\right) = -\frac{P}{3\epsilon_0}. \qquad (10.46)$$

As the direction of $\mathbf{P}$ was the only thing that distinguished the z axis, we can write our result in more general form:

$$\boxed{\mathbf{E}_{\text{in}} = -\frac{\mathbf{P}}{3\epsilon_0}} \qquad (10.47)$$

This is the macroscopic field $\mathbf{E}$ in the polarized material.

Figure 10.24 shows both the internal and external fields. At the upper pole of the sphere, the strength of the upward-pointing external field is, from Eq. (10.17) or Eq. (10.18) for the field of a dipole,

$$E_z = \frac{2p_0}{4\pi\epsilon_0 r^3} = \frac{2(4\pi r_0^3 P/3)}{4\pi\epsilon_0 r_0^3} = \frac{2P}{3\epsilon_0} \qquad \text{(outside, at top)}, \qquad (10.48)$$

which is just twice the magnitude of the downward-pointing internal field.

This example illustrates the general rules for the behavior of the field components at the surface of a polarized medium. $\mathbf{E}$ is discontinuous at the boundary of a polarized medium, exactly as it would be at a surface in vacuum that carried a surface charge density $\sigma = P_\perp$. The symbol $P_\perp$ stands for the component of $\mathbf{P}$ normal to the surface outward (which in the present case is $P_\perp = P\cos\theta$). It follows that $E_\perp$, the normal component of $\mathbf{E}$, must change abruptly by an amount $P_\perp/\epsilon_0$; whereas $E_\parallel$, the component of $\mathbf{E}$ parallel to the boundary, remains continuous, that is, has the same value on both sides of the boundary (Fig. 10.25). Indeed,

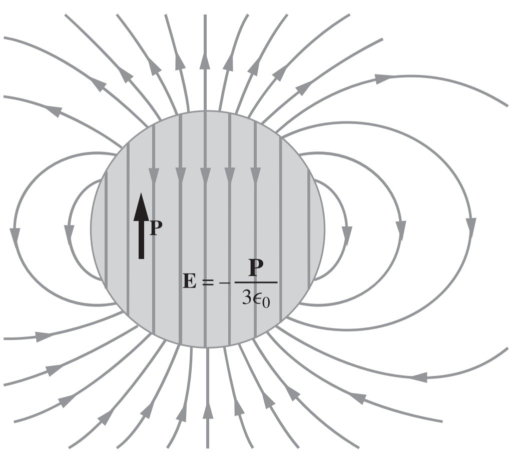

Figure 10.24.
The field of the uniformly polarized sphere, both inside and outside.

at the north pole of our sphere, the net change in E_z is $2P/3\epsilon_0 - (-P/3\epsilon_0)$, or P/ϵ_0.

Example (Continuity of $E_\parallel$) For our polarized sphere, let's check that the component of **E** parallel to the surface is continuous from inside to outside everywhere on the sphere. From Eq. (10.47) the internal field has magnitude $P/3\epsilon_0$ and points downward, so $E_\parallel^{\text{in}}$ is obtained by simply tacking on a factor of $\sin\theta$. That is, $E_\parallel^{\text{in}} = P\sin\theta/3\epsilon_0$. The tangential component of the external dipole field is given by the E_θ in Eq. (10.18):

$$E_\parallel^{\text{out}} = \frac{p_0 \sin\theta}{4\pi\epsilon_0 r^3} = \frac{(4\pi r_0^3 P/3)\sin\theta}{4\pi\epsilon_0 r_0^3} = \frac{P\sin\theta}{3\epsilon_0}, \tag{10.49}$$

which equals $E_\parallel^{\text{in}}$, as desired.

Note that, for $0 < \theta < \pi$, the $\sin\theta$ factor is positive, so $E_\parallel^{\text{in}}$ and $E_\parallel^{\text{out}}$ point in the positive $\hat{\boldsymbol{\theta}}$ direction, that is, away from the north pole. Similarly, for $\pi < \theta < 2\pi$, $E_\parallel^{\text{in}}$ and $E_\parallel^{\text{out}}$ point in the *negative* $\hat{\boldsymbol{\theta}}$ direction, which again is away from the north pole (because positive $\hat{\boldsymbol{\theta}}$ is directed clockwise around the full circle). A quick glance at Fig. 10.24 shows that the field lines are consistent with these facts.

The task of Exercise 10.36 is to use the explicit forms of the internal and external fields to show that $E_\perp$ has a discontinuity of $P_\perp/\epsilon_0$ everywhere on the surface of the sphere.

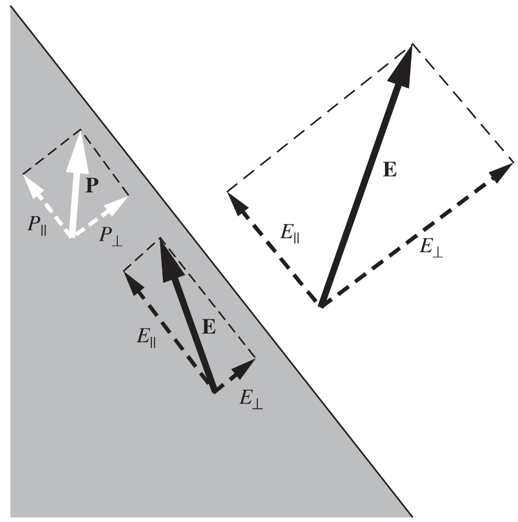

Figure 10.25.
The change in E at the boundary of a polarized dielectric: $E_\parallel$ is the same on both sides of the boundary; $E_\perp$ increases by $P_\perp/\epsilon_0$ in going from dielectric to vacuum. (Note that E and P/ϵ_0 are not drawn to the same scale.)

None of these conclusions depends on how the polarization of the sphere was caused. Assuming any sphere *is* uniformly polarized, Fig. 10.24 shows *its* field. Onto this can be superposed any field from other sources, thus representing many possible systems. This will not affect the discontinuity in **E** at the boundary of the polarized medium. The above rules therefore apply in any system, the discontinuity in **E** being determined solely by the existing polarization.

10.10 A dielectric sphere in a uniform field

As an example, let us put a sphere of dielectric material characterized by a dielectric constant κ into a homogeneous electric field $\mathbf{E}_0$ like the field between the parallel plates of a vacuum capacitor, Fig. 10.26. Let the sources of this field, the charges on the plates, be far from the sphere so that they do not shift as the sphere is introduced. Then whatever the field may be in the vicinity of the sphere, it will remain practically $\mathbf{E}_0$ at a great distance. This is what is meant by putting a sphere into a uniform field. The total field **E** is no longer uniform in the neighborhood of the sphere. It is the *sum* of the uniform field $\mathbf{E}_0$ of the distant sources and a field $\mathbf{E}'$ generated by the polarized matter itself:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'. \tag{10.50}$$

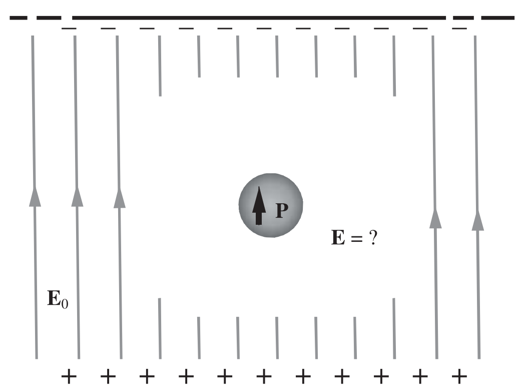

Figure 10.26.
The sources of the field $\mathbf{E}_0$ remain fixed. The dielectric sphere develops some polarization **P**. The total field **E** is the superposition of $\mathbf{E}_0$ and the field of this polarized sphere.

This relation is valid both inside and outside the sphere. The field $\mathbf{E}'$ depends on the polarization $\mathbf{P}$ of the dielectric, which in turn depends on the value of $\mathbf{E}$ inside the sphere:

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}_{\text{in}} = (\kappa - 1)\epsilon_0 \mathbf{E}_{\text{in}}. \tag{10.51}$$

Remember that the $\mathbf{E}$ that appears in this expression involving χ_e is the *total* electric field.

We don't know yet what the total field $\mathbf{E}$ is; we know only that Eq. (10.51) has to hold at any point inside the sphere. *If* the sphere becomes uniformly polarized, an assumption that will need to be justified by our results, the relation between the polarization $\mathbf{P}$ of the sphere and its own field at points inside, $\mathbf{E}'_{\text{in}}$, is given by Eq. (10.47):[6]

$$\mathbf{E}'_{\text{in}} = -\frac{\mathbf{P}}{3\epsilon_0}. \tag{10.52}$$

Substituting the $\mathbf{P}$ from Eq. (10.51) into Eq. (10.52) quickly gives $\mathbf{E}'_{\text{in}}$ in terms of $\mathbf{E}_{\text{in}}$; we obtain $\mathbf{E}'_{\text{in}} = -(\kappa - 1)\mathbf{E}_{\text{in}}/3$. Substituting this into Eq. (10.50) gives the total field inside the sphere as

$$\mathbf{E}_{\text{in}} = \mathbf{E}_0 - \frac{\kappa - 1}{3}\mathbf{E}_{\text{in}} \quad \Longrightarrow \quad \boxed{\mathbf{E}_{\text{in}} = \left(\frac{3}{2+\kappa}\right)\mathbf{E}_0} \tag{10.53}$$

Because κ is greater than 1, the factor $3/(2+\kappa)$ will be less than 1; the field inside the dielectric is weaker than $\mathbf{E}_0$. The polarization is

$$\mathbf{P} = (\kappa - 1)\epsilon_0 \mathbf{E}_{\text{in}} \quad \Longrightarrow \quad \boxed{\mathbf{P} = 3\left(\frac{\kappa - 1}{\kappa + 2}\right)\epsilon_0 \mathbf{E}_0} \tag{10.54}$$

The assumption of uniform polarization is now seen to be self-consistent.[7] To compute the total field $\mathbf{E}_{\text{out}}$ outside the sphere we must add vectorially to $\mathbf{E}_0$ the field of a central dipole with dipole moment equal to $\mathbf{P}$ times the volume of the sphere. Some field lines of $\mathbf{E}$, both inside and outside the dielectric sphere, are shown in Fig. 10.27.

To summarize, we found $\mathbf{E}_{\text{in}}$ by effectively equating two different expressions for the field $\mathbf{E}'_{\text{in}}$ caused by the polarized matter. One expression is simply the statement of superposition, $\mathbf{E}'_{\text{in}} = \mathbf{E}_{\text{in}} - \mathbf{E}_0$. The other expression is $\mathbf{E}'_{\text{in}} = -(\kappa - 1)\mathbf{E}_{\text{in}}/3$, which comes from the facts that $\mathbf{E}'_{\text{in}}$ is proportional to $\mathbf{P}$ (in the case of a sphere) and that $\mathbf{P}$ is proportional to $\mathbf{E}_{\text{in}}$ (in a linear dielectric).

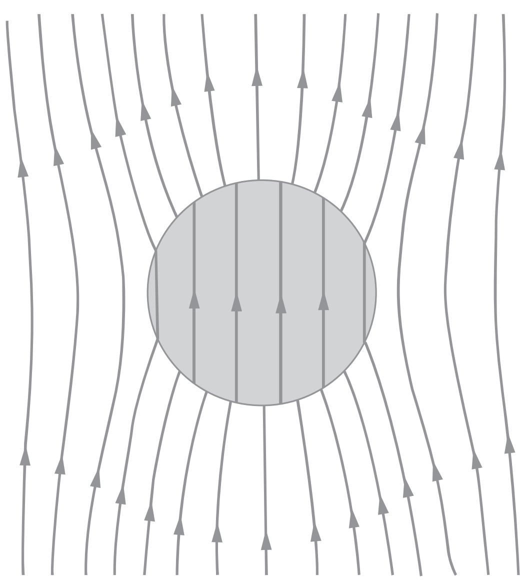

Figure 10.27.
The total field $\mathbf{E}$, both inside and outside the dielectric sphere.

[6] In Eq. (10.47) we were using the symbol $\mathbf{E}_{\text{in}}$, without the prime, for this field. In that case it was the only field present.

[7] That is what makes this system easy to deal with. For a dielectric cylinder of finite length in a uniform electric field, the assumption would not work. The field $\mathbf{E}'$ of a uniformly polarized cylinder – for instance one with its length about equal to its diameter – is *not* uniform inside the cylinder. (What must it look like?) Therefore $\mathbf{E}_{\text{in}} = \mathbf{E}_0 + \mathbf{E}'_{\text{in}}$ cannot be uniform – but in that case $\mathbf{P} = \chi_e \mathbf{E}_{\text{in}}$ could not be uniform after all. In fact, it is only dielectrics of ellipsoidal shape, of which the sphere is a special case, that acquire uniform polarization in a uniform field.

10.11 The field of a charge in a dielectric medium, and Gauss's law

Suppose that a very large volume of homogeneous linear dielectric has somewhere within it a concentrated charge Q, not part of the regular molecular structure of the dielectric. Imagine, for instance, that a small metal sphere has been charged and then dropped into a tank of oil. As was stated at the end of Section 10.8, the electric field in the oil is simply $1/\kappa$ times the field that Q would produce in a vacuum:

$$E = \frac{Q}{4\pi\epsilon_0\kappa r^2}. \tag{10.55}$$

The product $\epsilon_0\kappa$ is commonly denoted by ϵ, so we can write

$$\boxed{E = \frac{Q}{4\pi\epsilon r^2}} \qquad \text{where} \qquad \boxed{\epsilon \equiv \kappa\epsilon_0} \quad \Longrightarrow \quad \kappa = \frac{\epsilon}{\epsilon_0}. \tag{10.56}$$

The quantity ϵ is known as the *permittivity* of the dielectric. The vacuum permittivity, also called the *permittivity of free space,* is simply ϵ_0.

It is interesting to see how Gauss's law works out. The surface integral of $\mathbf{E}$ (which is the macroscopic, or space average, field, remember) taken over a sphere surrounding Q, gives $Q/\kappa\epsilon_0$, or Q/ϵ, if we believe Eq. (10.55), and *not* Q/ϵ_0. Why not? The answer is that Q is not the only charge inside the sphere. There are also all the charges that make up the atoms and molecules of the dielectric. Ordinarily any volume of the oil would be electrically neutral. But now the oil is radially polarized, which means that the charge Q, assuming it is positive, has pulled in toward itself the negative charge in the oil molecules and pushed away the positive charges. Although the displacement may be only very slight in each molecule, still on the average any sphere we draw around Q will contain more oil-molecule negative charge than oil-molecule positive charge. Hence the net charge in the sphere, including the "foreign" charge Q at the center, is *less* than Q. In fact, it is Q/κ.

It is often useful to distinguish between the foreign charge Q and the charges that make up the dielectric itself. Over the former we have some degree of control – charge can be added to or removed from an object, such as the plate of a capacitor. This is often called *free* charge. The other charges, which are integral parts of the atoms or molecules of the dielectric, are usually called *bound* charge. *Structural* charge might be a better name. These charges are not mobile; they are more or less elastically bound, contributing, by their slight displacement, to the polarization.

One can devise a vector quantity that is related by something like Gauss's law to the free charge only. In the system we have just examined (a point charge Q immersed in a dielectric), the vector $\kappa\mathbf{E}$ has this property. That is, $\int \kappa\mathbf{E}\cdot d\mathbf{a}$, taken over some closed surface S, equals Q/ϵ_0 if S encloses Q, and zero if it does not. By superposition, this must hold for any collection of free charges described by a free-charge density $\rho_{\text{free}}(x, y, z)$ in an infinite homogeneous linear dielectric medium:

$$\int_S \kappa \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_V \rho_{\text{free}} \, dv, \tag{10.57}$$

where V is the volume enclosed by the surface S. An integral relation like this implies a "local" relation between the divergence of the vector field $\kappa\mathbf{E}$ and the free charge density:

$$\text{div}\,(\kappa \mathbf{E}) = \frac{\rho_{\text{free}}}{\epsilon_0}. \tag{10.58}$$

Since κ has been assumed to be constant throughout the medium, Eq. (10.58) tells us nothing new. However, it can help us to isolate the role of the bound charge. In any system whatsoever, the fundamental relation (namely Gauss's law) between electric field $\mathbf{E}$ and total charge density $\rho_{\text{free}} + \rho_{\text{bound}}$ remains valid:

$$\text{div}\,\mathbf{E} = \frac{1}{\epsilon_0}(\rho_{\text{free}} + \rho_{\text{bound}}). \tag{10.59}$$

Subtracting Eq. (10.59) from Eq. (10.58) yields

$$\text{div}\,(\kappa - 1)\mathbf{E} = -\frac{\rho_{\text{bound}}}{\epsilon_0}. \tag{10.60}$$

According to Eq. (10.40), $(\kappa - 1)\mathbf{E} = \mathbf{P}/\epsilon_0$ for a linear dielectric, so Eq. (10.60) implies that

$$\boxed{\text{div}\,\mathbf{P} = -\rho_{\text{bound}}} \tag{10.61}$$

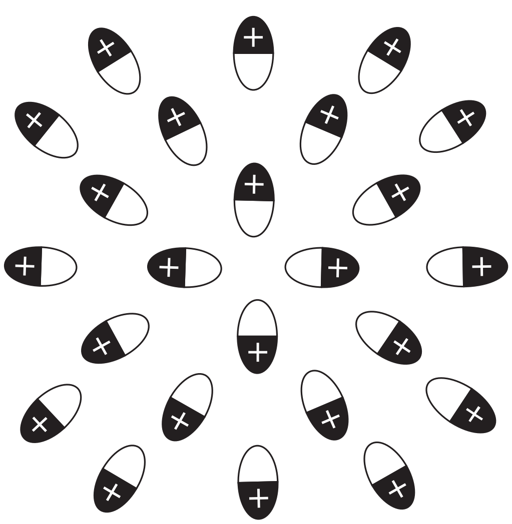

Figure 10.28.
Molecular dipoles arranged so that div $\mathbf{P} > 0$. Note the concentration of negative charge in the middle, consistent with Eq. (10.61).

Equation (10.61) states a local relation. It cannot depend on conditions elsewhere in the system, nor on how the particular arrangement of bound charges is maintained. Any arrangement of bound charge that has a certain local excess, per unit volume, of nuclear protons over atomic electrons must represent a polarization with a certain divergence. So, although we derived Eq. (10.61) by using relations pertaining to linear dielectrics, *it must in fact hold universally, not just in an unbounded linear dielectric.* It doesn't matter how the polarization comes about. (See Problem 10.11 for a general proof.) You can get a feeling for the identity expressed in Eq. (10.61) by imagining a few polar molecules arranged to give a polarization with a positive divergence (Fig. 10.28). The dipoles point outward, which necessarily leaves a little concentration of negative charge in the middle. Of course, Eq. (10.61) refers to averages over volume elements so large that $\mathbf{P}$ and ρ_{bound} can be treated as smoothly varying quantities.

From Eqs. (10.59) and (10.61), both of which are true in any system whatsoever, we get the relation

$$\text{div}\,(\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\text{free}}. \tag{10.62}$$

This is quite independent of any relation between **E** and **P**; it is not limited to linear dielectrics (where **P** is proportional to **E**).

It is customary to give the combination $\epsilon_0\mathbf{E}+\mathbf{P}$ a special name, the *electric displacement* vector, and its own symbol, **D**. That is, we define **D** by

$$\boxed{\mathbf{D} \equiv \epsilon_0\mathbf{E}+\mathbf{P}} \tag{10.63}$$

and Eq. (10.62) becomes

$$\boxed{\mathrm{div}\,\mathbf{D} = \rho_{\text{free}}} \tag{10.64}$$

This relation, or equivalently Eq. (10.62), holds in any situation in which the macroscopic quantities **P**, **E**, and ρ can be defined.

If additionally we are dealing with a linear dielectric, then by comparing Eqs. (10.58) and (10.64) we see that **D** is simply $\kappa\epsilon_0\mathbf{E}$, or

$$\boxed{\mathbf{D} = \epsilon\mathbf{E}} \qquad \text{(for a linear dielectric)}. \tag{10.65}$$

This alternatively follows from Eq. (10.63) by using Eq. (10.41) to write **P** as $\chi_e\epsilon_0\mathbf{E}$, and then using Eq. (10.42) to write $1+\chi_e$ as κ.

The appearance of Eq. (10.64) may suggest that we should look on **D** as a vector field whose source is the free charge distribution ρ_{free} (up to a factor of ϵ_0), in the same sense that the total charge distribution ρ is the source of **E**. That would be wrong. The electrostatic field **E** is uniquely determined – except for the addition of a constant field – by the charge distribution ρ because, supplementing the law $\mathrm{div}\,\mathbf{E} = \rho/\epsilon_0$, there is another universal condition, $\mathrm{curl}\,\mathbf{E} = 0$. It is *not* true, in general, that $\mathrm{curl}\,\mathbf{D} = 0$. Thus the distribution of free charge is not sufficient to determine **D** through Eq. (10.64). Something else is needed, such as the boundary conditions at various dielectric surfaces. The boundary conditions on **D** are of course merely an alternative way of expressing the boundary conditions involving **E** and **P**, already stated near the end of Section 10.9 and in Fig. 10.25.

Example (Continuity of $D_\perp$) For our polarized sphere in Section 10.9, we saw that $E_\parallel$ was continuous across the boundary whereas $E_\perp$ was not. These boundary conditions hold for any shape of polarized material. It turns out that the opposite conditions are true for **D**. That is, $D_\perp$ is continuous across the boundary whereas $D_\parallel$ is not. You can derive these boundary conditions in Problem 10.12. For now, let's just verify that $D_\perp$ is continuous across the boundary of our polarized sphere.

Inside the sphere, we have $\mathbf{E} = -\mathbf{P}/3\epsilon_0$, so the displacement vector is $\mathbf{D} = \epsilon_0(-\mathbf{P}/3\epsilon_0)+\mathbf{P} = 2\mathbf{P}/3$. The radial component of this is

$$D_\perp^{\text{in}} \equiv D_r^{\text{in}} = \frac{2P\cos\theta}{3}. \tag{10.66}$$

Outside the sphere, **E** is the field due to a dipole with $\mathbf{p}_0 = (4\pi R^3/3)\mathbf{P}$. The radial component of the dipole field is $E_r = p_0\cos\theta/2\pi\epsilon_0 R^3$. In terms of P this becomes $E_r = 2P\cos\theta/3\epsilon_0$. Since $\mathbf{P} = 0$ outside the sphere, the external $\mathbf{D}$ is obtained by simply multiplying the external $\mathbf{E}$ by ϵ_0. Therefore

$$D_{\perp}^{\text{out}} \equiv D_r^{\text{out}} = \frac{2P\cos\theta}{3}. \tag{10.67}$$

This equals the above $D_{\perp}^{\text{in}}$, as desired.

The task of Exercise 10.41 is to use the explicit forms of the internal and external fields to find the discontinuity in $D_{\|}$ everywhere on the surface of the sphere.

In the approach we have taken to electric fields in matter, the introduction of $\mathbf{D}$ is an artifice that is not, on the whole, very helpful. We have mentioned $\mathbf{D}$ because it is hallowed by tradition, beginning with Maxwell,[8] and the student is sure to encounter it in other books, many of which treat it with more respect than it deserves.

Our essential conclusions about electric fields in matter can be summarized as follows:

(1) Matter can be polarized, its condition being described completely, so far as the macroscopic field is concerned, by a polarization density $\mathbf{P}$, which is the dipole moment per unit volume. The contribution of such matter to the electric field $\mathbf{E}$ is the same as that of a charge distribution ρ_{bound}, existing in vacuum and having the density $\rho_{\text{bound}} = -\text{div}\,\mathbf{P}$. In particular, at the surface of a polarized substance, where there is a discontinuity in $\mathbf{P}$, this reduces to a surface charge of density $\sigma = -\Delta P_{\perp}$. Add any free charge distribution that may be present, and the electric field is the field that this *total* charge distribution would produce in vacuum. This is the macroscopic field $\mathbf{E}$ both inside and outside matter, with the understanding that inside matter it is the spatial average of the true microscopic field.

(2) If $\mathbf{P}$ is proportional to $\mathbf{E}$ in a material, we call the material a linear dielectric. We define the electric susceptibility χ_e and the dielectric constant κ characteristic of that material as $\chi_e = \mathbf{P}/\epsilon_0\mathbf{E}$ and $\kappa = 1 + \chi_e$. Free charges immersed in a linear dielectric give rise to electric fields that are $1/\kappa$ times as strong as the same charges would produce in vacuum.

10.12 A microscopic view of the dielectric

The polarization $\mathbf{P}$ in the dielectric is simply the large-scale manifestation of the electric dipole moments of the atoms or molecules of which

[8] The prominence of **D** in Maxwell's formulation of electromagnetic theory, and his choice of the name *displacement*, can perhaps be traced to his inclination toward a kind of mechanical model of the "aether." Whittaker has pointed out in his classic text (Whittaker, 1960) that this inclination may have led Maxwell himself astray at one point in the application of his theory to the problem of reflection of light from a dielectric.